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Nonlinear dynamics of vorticity waves in the coastal zone

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Vorticity waves are wave-like motions occurring in various types of shear flows. We study the dynamics of these motions in alongshore shear currents in situations where it can be described within weakly nonlinear asymptotic theory. The principal mechanism of vorticity waves can be interpreted as potential vorticity conservation with the background vorticity gradient provided both by the mean current shear and the variation of depth. Under the assumption that the mean potential vorticity distibution is monotonic in the cross-shore direction, the nonlinear stage of the dynamics of weakly nonlinear vorticity waves, long in comparison with the current cross-shore scale, is found to be governed by an evolution equation of the generalized Benjamin–Ono type. The dispersive terms are given by an integro-differential operator with the kernel determined by the large-scale cross-shore depth and current dependence. The derived equations form a wide new class of nonlinear evolution equations. They all tend to the Benjamin–Ono equation in the short-wave limit, while in the long-wave limit their asymptotics depend on the specific form of the depth and current profiles. For a particular family of model bottom profiles the equations are 'intermediate' between Benjamin–Ono and Korteweg–de Vries equations, but are distinct from the Joseph intermediate equation. Solitary-wave solutions to the equations for these depth profiles are found to decay exponentially. Taking into account coastline inhomogeneity or/and alongshore depth variations adds a linear forcing term to the evolution equation, thus providing an effective generation mechanism for vorticity waves.

1. Introduction

Vorticity waves are wave-like motions occurring in various types of shear flows and characterized by one common feature: the restoring force providing oscillations of the fluid particles and through that causing the motions under consideration is due to the gradient of vorticity of the basic flow (see Lin 1955). While the evolution of infinitesimal perturbations in shear flows can always be naturally described in terms of wave-like normal modes (e.g. Lin 1955; Drazin & Reid 1981), finite-amplitude disturbances might be treated as waves only under some special conditions. One of the most typical difficulties lies in the fact that very often instabilities are so strong (say in shear flows with inflection points) that the description of the perturbation evolution in terms of weakly nonlinear theories makes no particular sense. To describe strongly nonlinear motions it is sometimes more natural to use a vortex dynamics description (see e.g. Saffman 1993) but most often no convenient description exists. Another typical obstacle occurring even for the weakly nonlinear motions is caused by the singular character of the linear eigenmodes of a continuous spectrum, with the straightforward perturbation approach no longer working as all the products due to nonlinear terms diverge. To our knowledge no universal cure for this problem has yet been developed. Thus the physical situations where a self-consistent analysis of weakly nonlinear vorticity waves is possible, their dynamics is non-trivial and these waves are the dominant modes of motions, are of special interest.

The evolution of vorticity waves of finite amplitude satisfying the conditions formulated above has been studied in very different contexts, for example as secondary two-dimensional structures in Blasius boundary layers (e.g. Kachanov, Ryzhov & Smith 1993) and in plane Poiseuille flow (Pedley & Stephanoff 1985) and as three-dimensional waves in boundary layers in water on air–water interface (Shrira 1989). We note that, though being weakly nonlinear, vorticity waves may have a shape essentially distinct from sinusoidal. The geophysical situations when these waves are the dominating type of motion merit special consideration.

The main questions we address in this paper might be formulated as follows: Is there a place for motions of this type in the coastal zone? If yes, in what geophysical situations can they occur? What equations govern their evolution and what physics lie behind the equations? We are particularly interested in understanding the interplay of nonlinear and dispersive effects which commonly provide most of the richness of nonlinear wave dynamics.

In recent years the term 'vorticity waves' in the context of geophysical fluid dynamics was applied mainly to the nearshore motions first found in the 'SUPERDUCK' field experiment by Oltman-Shay, Howd & Birkemeier (1989). These relatively small-scale motions occur in the surf-zone alongshore shear currents which produce a vorticity field with non-monotonic cross-shelf dependence. They were interpreted as the linearly unstable modes within the linear Rayleigh-type boundary-value problem by Bowen & Holman (1989) (see also more recent works by Dodd 1994 and Falques & Iranzo 1994 and the references therein for the extension to realistic models of the mean current and bottom profile with bottom friction taken into account). The basic potential vorticity distribution must be non-monotonic for the linear instability to appear. However a weakly nonlinear analysis of Shrira, Voronovich & Kozhelupova (1996) showed that an even more intense mechanism of vorticity wave generation, that of explosive instability, is likely to occur in such currents. This makes questionable the possibility of an adequate description in terms of *waves* of the later stages of their evolution, within either linear or weakly nonlinear wave theory. One may expect a description in terms of interacting vortices to be more promising. In the present study we exclude motions of such a type from consideration and concentrate upon 'waves'.

It should be mentioned here again that the term 'vorticity waves' is used in a general sense for *all* wave-like perturbations of the alongshore currents provided by the potential vorticity conservation mechanism, *not* for the linearly unstable modes alone. It is to be stressed that the neutral modes whose nonlinear evolution is considered in the present paper, as well as weakly decaying modes, can be of interest and importance for coastal-zone dynamics. We shall study vorticity waves in the general sense specified above in the coastal zone, focusing our attention upon the motions *a priori* expected to be properly described as *waves* within weakly nonlinear theory, i.e. we investigate the evolution of finite-amplitude wave-like perturbations to alongshore currents. This means in particular that we confine ourselves to considering monotonic potential vorticity profiles. The motions under study may differ in scales greatly although they remain relatively small-scale ones not to be strongly influenced by Coriolis forces. We mention here some examples of relevant geophysical contexts:

(*a*) strong Western boundary currents sometimes come very close to the shore producing a narrow zone of strong vorticity gradient;

(*b*) tidal or other currents in straits are often characterized by very narrow (in comparison to the strait width) boundary layers (e.g. Defant 1961, p.189);

(*c*) currents produced by breaking waves near a wall or very steep shore.

The paper is organized as follows. We start in § 2 with the standard shallow-water equations and retaining the leading-order terms in the Froude number arrive at a version of the vorticity equation. The boundary problem comprising this nonlinear equation and the zero-flux boundary conditions on the shore and at infinity forms the mathematical framework of our study. Confining ourselves to consideration of the nearshore with a monotonic distribution of the basic potential vorticity and assuming the motions under study to be: (i) long in comparison with characteristic scales of the current cross-shore variation and the 'fast' depth scale; (ii) weakly nonlinear; we employ the multiple-scale method to simplify our nonlinear boundary problem.

In § 3 an asymptotic technique very similar to that used in a different context by Shrira (1989) is applied to derive a nonlinear evolution equation. The equation takes into account at the leading-order nonlinearity and dispersion, the specific form of the pseudo-differential dispersion operator being determined by the 'slow' cross-shore depth and/or current dependence. Thus it is better to speak about a 'class of evolution equations' with specific equations corresponding to different depth and current profiles. For a family of model cross-shore depth dependencies the derived equations are found to be 'intermediate' between classical Benjamin–Ono and Korteweg–de Vries (KdV) equations.

In § 4 some universal asymptotic properties of solitary-wave solutions are found analytically for the 'intermediate equations: the 'tails' of solitary waves decay exponentially like those of KdV solitons. The complete solitary-wave profiles, as an example, for a particular class of depth profiles, namely exponential, are obtained numerically.

In § 5 we are concerned with the influence of 'weak' Earth's rotation: the Coriolis force is assumed to be small, but not negligible as in § 3, in comparison with the mean vorticity forces. The evolution equation, which takes the Coriolis effect into account, and so differing from that of the previous section in the coefficients and the specific form of the kernel in the dispersion operator, has been derived. The linear resonance between vorticity waves and continental shelf waves is briefly discussed. When there is resonance the dispersion operator becomes singular and the evolution equation loses its validity.

In § 6 we study the generation of vorticity waves by an uneven coastline, aiming to estimate the effectiveness of this mechanism. This part of our work is strongly influenced by the work of Grimshaw (1987), where resonant forcing of barotropic continental shelf waves was studied under similar assumptions. It is worth mentioning, however, that though both vorticity and continental-shelf waves are driven by the forces due to the gradient of potential vorticity and in this sense are similar to each other, still they have essential differencies in their physics. Continental-shelf waves exist mainly due to the gradient in the potential vorticity field created by the Earth's rotation and varying depth of the fluid. They are only *influenced* by the currents and can exist in a rotating fluid without currents. On the other hand, vorticity waves appear owing to a potential vorticity field supplied by a shear current, and they can exist even in a non-rotating fluid of constant depth. The presence of a critical layer is also essential for the waves we study, while Grimshaw (1987) excluded situations with critical layers from consideration. To understand better the nature of the difference between these two types of motions it is helpful to consider first the simplest piecewise-constant model of the nearshore with a single jump in the profile of potential vorticity due to a shear current. Then one would obtain all the continental-shelf wave modes slightly affected by the current *plus* an additional wave on the vorticity jump. This additional mode is our vorticity wave. If the jump is smoothed the *discrete* spectrum mode on this jump disappears. Thus from the mathematical point of view the vorticity waves are the *intermediate asymptotics* of the packets of *continous* spectrum modes (see Shrira 1989), while continental-shelf waves are the true discrete spectrum modes.

In § 7 we briefly summarize the main results of the study and discuss the key assumptions, limitations and perspectives of the approaches developed.

2. Basic equations and scaling

We shall consider finite-amplitude vorticity wave dynamics in the coordinate frame with the axes x and y directed offshore and alongshore respectively. Being interested in studies of motions with periods much smaller than the Coriolis time scale† we first neglect the influence of the Earth's rotation on the vorticity wave dynamics. The total velocity field is assumed to consist of the mean alongshore current with a cross-shore shear and perturbations

$$\boldsymbol{u}_* = \{u(x,y,t),\ V(x) + v(x,y,t)\}, \tag{2.1}$$

where $V(x)$ represents the mean steady current and $\boldsymbol{u} = \{u, v\}$ the perturbed velocity field. The basic equations governing nearshore vorticity wave dynamics are then the standard shallow-water equations

$$u_t + Vu_y = -g\zeta_x - (uu_x + vu_y), \tag{2.2a}$$

$$v_t + Vu_y + uV_x = -g\zeta_y - (uv_x + vv_y), \tag{2.2b}$$

$$\zeta_t + V\zeta_y + [(\zeta + h)u]_x + [(\zeta + h)v]_y = 0. \tag{2.2c}$$

Here ζ is the free-surface elevation, g is gravity acceleration, $h = h(x)$ is the depth presumed to be alongshore uniform.

The scalings

$$\left.\begin{aligned} u &= \frac{d}{L}V_0 u'; & v &= V_0 v'; & V &= V_0 V'; & \zeta &= \frac{V_0^2}{g}\zeta'; \\ h &= h_0 h'; & x &= d\,x'; & y &= L\,y'; & t &= \frac{L}{V_0}t'; \end{aligned}\right\} \tag{2.3}$$

where V_0 is the typical magnitude of the mean current velocity, say its maximum nearshore value, d is the typical mean current cross-shore variability scale, L is the typical wavelength, h_0 is the typical depth within the current domain (the primed quantities are non-dimensional), yield the non-dimensional equations of motion in the form

$$u_t + Vu_y = -\epsilon^{-2}\zeta_x - (uu_x + vu_y), \tag{2.4a}$$

$$v_t + Vu_y + uV_x = -\zeta_y - (uv_x + vv_y), \tag{2.4b}$$

$$(hu)_x + (hv)_y = 0. \tag{2.4c}$$

† Typically the Coriolis period is one or two orders larger and thus the Coriolis effects could as a rule be neglected. The situations where these effects are comparable with some other small factors and therefore should be taken into account will be considered below in § 5.

Here the parameter ϵ represents the ratio of the cross- and alongshore spatial scales

$$\epsilon = \frac{d}{L} \tag{2.5}$$

and the primes have been dropped for convenience. The terms containing surface elevation, which are of the second order in the Froude number $\mathscr{F} = V_0/(gh_0)^{1/2}$, were neglected in the conservation of mass relation (2.4*c*) owing to the smallness of $\mathscr{F}$. The validity of this assumption†, as well as neglect of the Coriolis effect in comparison with the inertial terms involving the mean alongshore current, for the description of vorticity wave dynamics can be easily checked.

Equation (2.4*c*) allows us to introduce a stream function $\psi(x, y, t)$ such that

$$hu = -\psi_y, \qquad hv = \psi_x. \tag{2.6}$$

Cross-differentiating (2.4*a, b*) to remove the surface elevation and substituting the stream function (2.6) into the result one easily gets the nonlinear vorticity equation

$$(\partial_t + V\partial_y)\left(\left(\frac{\psi_x}{h}\right)_x + \epsilon^2\frac{\psi_{yy}}{h}\right) - \left(\frac{V'}{h}\right)'\psi_y = \partial_x\left(-\frac{\psi_x}{h}\frac{\psi_{xy}}{h} + \frac{\psi_y}{h}\left(\frac{\psi_x}{h}\right)_x\right)$$
$$+\epsilon^2\partial_y\left(\frac{\psi_y}{h}\left(\frac{\psi_y}{h}\right)_x - \frac{\psi_x}{h}\frac{\psi_{yy}}{h}\right). \tag{2.7}$$

The appropriate boundary conditions for this problem are the requirements of zero mass fluxes through the coastline $x = x_0(y)$ and at infinity:

$$\psi_y + (hV + \psi_x)\,\partial_y x_0 = 0 \qquad \text{at} \quad x = x_0(y), \tag{2.8a}$$

$$\psi_y = 0 \qquad \text{as} \quad x \to \infty. \tag{2.8b}$$

For a straight coast, $x_0(y)$ without loss of generality can be put equal to zero. Unless otherwise stated we shall focus our attention on this case. The essentially nonlinear boundary-value problem (2.7), (2.8) has no regular way of being treated. For an arbitrary shore-zone profile and current structure a few *a priori* bounds can be derived in a regular manner for the complex phase speed in the linearized problem only, these being straightforward‡ generalizations of the classical results of the hydrodynamic stability theory (see e.g. Drazin & Reid 1981) based on the Rayleigh equation. To get a quantitative description rather than just bounds the authors of all the previous works, even within the linearized problem, were forced to deal either with specific piecewise models or to use a numerical treatment.

We aim to study analytically the nonlinear dynamics of vorticity waves. The specific goals are to simplify the description of weakly nonlinear motions by making use of the relevant small parameters, to study the role of forcing by shore and bottom inhomogeneities and of 'slow' Earth rotation on their dynamics. So we confine our attention to consideration of the shore zones characterized by the monotonic cross-shore distribution of the mean potential vorticity, i.e.

$$\left(\frac{V'}{h}\right)' \neq 0 \quad \forall\, x. \tag{2.9}$$

This condition means that the potential vorticity profile has no extremal points, is

† It is often called the non-divergent approximation and is widely used in geophysical hydrodynamics (e.g. Mei 1993).

‡ In particular, the bounds on the instability characteristics can be deduced from the results by Collings & Grimshaw (1984) just by putting Coriolis parameter equal to zero.

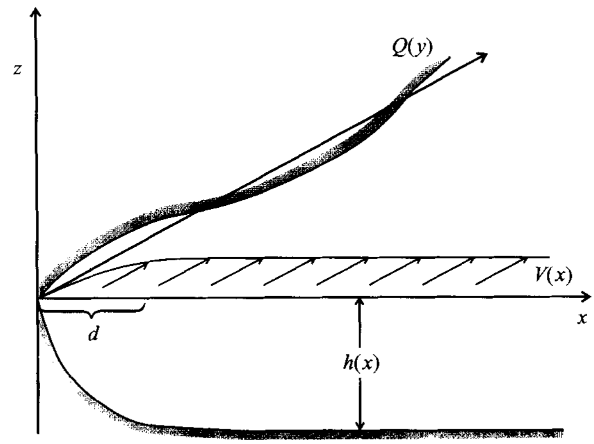

FIGURE 1. Geometry and notation.

analogous to that of the absence of inflection points for the Rayleigh equation and is imposed to ensure linear stability and to make the technique similar to that developed in Shrira (1989), Grimshaw (1987) applicable. Meanwhile, it does not imply that the mean current velocity and the depth profile be separately monotonic.

For definiteness we consider mainly the mean current equating to zero at the coast and rapidly tending to some constant value V_∞ as the cross-shore coordinate x tends to infinity (figure 1), the exact sense of the world 'rapidly' being defined below. While the characteristic cross-shore scale of the current is designated d, the depth profile is considered as having two cross-shore scales: 'fast', of order d and 'slow', the much greater scale d_{slow}, i.e. $h(x)$ is taken in the form

$$h(x) = H(x)D(\mu\epsilon x) \qquad \text{where} \quad \mu = \frac{L}{d_{slow}}. \tag{2.10}$$

The two-scale mean currents will be considered below as well.

Perturbations of the mean flow are assumed to be small but finite and a corresponding nonlinearity parameter ϵ_n is defined as the ratio of the typical value of the alongshore velocity perturbation v to the maximal mean current velocity V_∞:

$$\epsilon_n = \frac{v}{V_\infty}. \tag{2.11}$$

Thus the problem is governed by three non-dimensional parameters: μ, ϵ and ϵ_n. To proceed further we have to presume some kind of balance between them. We shall study motions with alongshore scale L much larger than the mean current cross-shore scale d, presume the balance

$$\epsilon_n = \epsilon \ll 1 \tag{2.12}$$

and consider μ to be $O(1)$, at least for a while, to have a single 'slow' spatial scale. Thus, there exists a natural separation of scales in the problem, which prompts one to use the multiple-scale method, i.e. introduce a set of spatial and temporal variables

$$\xi = x; \qquad X = \epsilon x; \qquad Y = (y - ct); \qquad T = \epsilon t. \tag{2.13}$$

First, for simplicity only, we continue to consider the mean flow profiles without any slow cross-shore dependence, while the depth profile depends both on 'fast' and 'slow' scales, i.e.

$$h = H(\xi)D(X); \qquad V = V(\xi). \tag{2.14}$$

In terms of the new variables the primary differential operators become

$$\partial_x = \partial_\xi + \epsilon\partial_X; \qquad \partial_y = \partial_Y; \qquad \partial_t = -c\partial_Y + \epsilon\partial_T. \tag{2.15}$$

Boundary conditions (2.8*a, b*) are now applied to the function which depends on the two cross-shore variables and thus should be modified to be

$$\psi_Y\Big|_{\xi=0} = 0, \tag{2.16a}$$

$$\psi_Y \to 0 \quad \text{as} \quad X \to \infty, \tag{2.16b}$$

$$\partial^2_{\xi Y}\psi\Big|_{X=0} \to 0 \quad \text{as} \quad \xi \to \infty. \tag{2.16c}$$

The nonlinear vorticity equation (2.7) plus the modified boundary conditions (2.16*a–c*) form the principal framework of our study.

3. Nonlinear evolution equation

3.1. *The asymptotic derivation*

According to the standard procedure (e.g. Nayfeh 1973) we look for a solution $\psi(\xi, X, Y, T)$ of the nonlinear boundary problem (2.7), (2.16) in the form of an asymptotic series in powers of ϵ:

$$\psi = \sum_{n=1}^{\infty} \epsilon^n \psi_n(\xi, X, Y, T). \tag{3.1}$$

To this end we substitute (3.1) into (2.7) and consider terms of the same order in ϵ. At the main (first) order one obtains

$$\partial_Y \mathsf{L}[\psi_1] = 0, \tag{3.2}$$

where L is an ordinary differential operator of the form

$$\mathsf{L}[\psi] = (V-c)\left(\frac{\psi'}{H}\right)' - \left(\frac{V'}{H}\right)'\psi = \left(\frac{(V-c)^2}{H}\left(\frac{\psi}{V-c}\right)'\right)' \tag{3.3}$$

and a prime designates the derivative with respect to ξ. We are looking for wave-like solutions propagating in the alongshore direction, which implies that the stream function cannot be constant with respect to Y. So instead of (3.2) we have from the first-order approximation

$$\mathsf{L}[\psi_1] = 0. \tag{3.4}$$

This equation with modified boundary conditions determines the boundary value problem for the function ψ_1 and the appropriate eigenvalue c. Equation (3.4) is readily integrable, its two independent solutions being

$$\left.\begin{aligned} \psi_1^{(1)} &= (V-c)(\phi * A), \\ \psi_1^{(2)} &= (\phi^{(2)} * A^{(2)})(V-c)\int_{\xi_0}^{\xi} \frac{H\,\mathrm{d}\xi'}{(V-c)^2}, \end{aligned}\right\} \tag{3.5}$$

where the functions $\phi = \phi(X,Y)$ and $A = A(T,Y)$ are introduced to separate the temporal and 'slow' cross-shore dependence of the solution, while the notation $(f * g)$ designates a convolution of two functions with respect to the alongshore coordinate:

$$(f * g) = \int_{-\infty}^{+\infty} f(X, Y') g(T, Y - Y') \, \mathrm{d}Y'. \tag{3.6}$$

The second fundamental solution $\psi_1^{(2)}$ diverges as ξ tends to infinity and cannot satisfy (2.16*c*) (as $\partial_Y A^{(2)}$ does not equal zero identically). Applying (2.16*a*) to the first solution we find the celerity of the vorticity wave:

$$c = V\big|_{\xi=0} = 0 \tag{3.7}$$

as well as the constraint on the slowly varying part of the solution $\phi(X,Y)$:

$$\phi(X,Y) \to 0 \qquad \text{as} \qquad X \to \infty. \tag{3.8}$$

The result (3.7) indicates that vorticity waves within the frame of our adopted scaling are slow-evolving, i.e. they are steady at the main-order approximation and their evolution occurs on the 'slow' timescale specified by the next-order effects of nonlinearity and dispersion.

At the second order in ϵ, (2.7) together with the results of the first order yields an inhomogeneous differential equation for the second-order correction ψ_2 to the stream function with the right-hand side expressed in terms of the first-order solutions:

$$\partial_Y \mathsf{L}[\psi_2] = -(\phi * A_T)\left(\frac{V'}{H}\right)' - (\phi_X * A_Y)\left(\frac{V^2}{H}\right)' \\ +(\phi * A)(\phi * A_Y) D^{-1}\left(\frac{V^2}{H}\left(\frac{V'}{H}\right)'\right)'. \tag{3.9}$$

Integrating (3.9) twice with respect to ξ and again using (2.16*b, c*) one readily gets

$$\partial_Y \psi_2 = (\phi * A_T) + (\phi_X * A_Y) V \int_{\infty}^{\xi} \left(\frac{V_\infty^2}{V^2(\zeta)} \frac{H(\zeta)}{H_\infty} - 1\right) \mathrm{d}\zeta + (\phi * A)(\phi * A_Y) D^{-1} \frac{V'}{H}. \tag{3.10}$$

Here $H_\infty \equiv \lim_{\xi \to \infty} H(\xi)$, the slowly varying part of the depth profile should not be zero anywhere and the integral in (3.10) should exist. This last condition specifies the class of mean flows and fast-varying depth cross-shore dependencies to which our approach could be applied.

In turn (2.16*a*) when applied to (3.10) leads to an equality valid at the coast $\xi = 0$:

$$(\phi * A_T) - \frac{V_\infty^2}{H_\infty} \frac{H}{V'} (\phi_X * A_Y) + D^{-1} \frac{V'}{H} (\phi * A)(\phi * A_Y) = 0. \tag{3.11}$$

We suppose that the depth and the mean current velocity derivative either both do not equal zero at the coast, i.e. at $\xi = 0$, or both equal zero simultaneously thus requiring the nearshore background vorticity field to have no singularities or zeros, which seems to be quite reasonable. So the quotient V'/H at $\xi = 0$ should be regarded as the limit when ξ tends to zero, that is, if both V' and H equal zero at the coast

$$\left.\frac{V'}{H}\right|_{\xi=0} = \lim_{\xi \to 0} \frac{V'(\xi)}{H(\xi)} = \left.\frac{V''}{H'}\right|_{\xi=0}. \tag{3.12}$$

To determine the solution dependence on the slow cross-shore variable we have to explore the next (third-order) approximation. As is common for the multiple-scale method, in this approximation one gets again an inhomogeneous differential equation with the same homogeneous part and the right-hand side containing both secular and non-secular terms. Secularity in this context means non-integrablilty of the corresponding terms with respect to the fast cross-shore variable. Therefore for the third-order term to be regular, the secular terms must equal zero identically. This requirement together with (3.8) yields the boundary problem for the slowly varying function $\phi(X, Y)$:

$$\left(\left(\frac{\phi_X}{D}\right)_X * A\right) + \frac{1}{D}(\phi * A_{YY}) = 0, \tag{3.13a}$$

$$\phi(X, Y) \to 0 \quad \text{at} \quad X \to \infty. \tag{3.13b}$$

It is convenient now to present the first-order stream function ψ_1 in the Fourier form

$$\psi_1 = V \int_{-\infty}^{+\infty} \phi_k(X) A_k(T) \exp\{-ikY\} \, dk \tag{3.14}$$

and to define the Fourier amplitude $A_k(T)$ so that

$$\tilde{A}_k(T) = \phi_k(0) A_k(T). \tag{3.15}$$

Eventually, after substituting (3.14) into (3.11), (3.13) and performing direct and inverse Fourier transforms of these equalities, one obtains the nonlinear evolution equation governing the dynamics of weakly nonlinear, weakly dispersive vorticity waves (the tilde sign is omitted)

$$\partial_T A + sA\partial_Y A - r\,\mathsf{G}[\partial_Y A] = 0, \quad r = \left.\frac{V_\infty^2}{H_\infty}\frac{H}{V'}\right|_{\xi=0}, \qquad s = \left.\frac{V'}{H}\right|_{\xi=0}, \tag{3.16}$$

where the pseudo-differential operator $\mathsf{G}[f]$ is prescribed by the slow cross-shore variation of the depth:

$$\mathsf{G}[f] = \iint_{-\infty}^{+\infty} \left.\frac{\partial_X \phi_k}{\phi_k}\right|_{X=0} f(Y') \exp\{ik(Y - Y')\} \, dk \, dY' \tag{3.17}$$

and $\phi_k(X)$ is the solution of the following boundary problem:

$$\partial_X \left(\frac{\partial_X \phi_k}{D}\right) - k^2 \frac{\phi_k}{D} = 0, \tag{3.18a}$$

$$\phi_k(X) \to 0 \quad \text{as} \quad X \to \infty. \tag{3.18b}$$

The separation of the depth cross-shore dependence into 'fast' and 'slow' parts was performed so that $D(0) = 1$.

3.2. *A critical layer problem*

The above theory has an apparent shortcoming. The derived asymptotic solution is not uniformly convergent near the coast because the value of the alongshore velocity perturbation there becomes comparable with that of the mean current. Moreover, the second- and third-order terms in (3.1) become singular at the point $\xi = 0$. Evidently, there exists a critical layer of width $O(\epsilon)$ near the coast where the vorticity wave phase speed coincides with that of the mean current (according to (3.7)). Inside the critical layer (3.1) is not valid any more since the motion seems to be strongly nonlinear and the viscous effects are likely to play an important role. However, there is a remarkable fact that allows us to overcome this difficulty. Namely, to $O(\epsilon)$ the critical layer does not influence the dynamics of vorticity waves outside it. Therefore, the nonlinear dynamics of vorticity waves outside the critical layer is governed by (3.16). This fact can be proved by setting

$$\tilde{z} = \frac{\xi}{\epsilon}, \qquad \psi = \epsilon^2 \tilde{\psi}, \qquad V = \epsilon V' \tilde{z} + o(\epsilon) \tag{3.19}$$

and calculating the main-order terms of the inner expansion of the solution (3.5), (3.10). We easily get (for $H(0), V'(0) \neq 0$)

$$\tilde{\psi} \to V' A \tilde{z} + \partial_Y^{-1} \partial_T A + s\tfrac{1}{2} A^2 - r\mathrm{G}[A] + O(\epsilon \ln \epsilon). \tag{3.20}$$

The boundary condition then leads to (3.16) as the $O(\epsilon \ln \epsilon)$ term tends to zero as $\epsilon \to 0$. Finding a solution inside the critical layer represents a much more complicated problem and is beyond the scope of the present paper.

A similar result holds when both $V'(0)$ and $H(0)$ equal zero. A simple consideration indicates that in this case the critical layer width is $O(\epsilon^{1/2})$ and therefore we introduce new inner variables

$$\hat{z} = \frac{\xi}{\epsilon^{1/2}}, \qquad \psi = \epsilon^2 \hat{\psi}, \qquad V = \epsilon \tfrac{1}{2} V'' \hat{z}^2 + o(\epsilon). \tag{3.21}$$

The main-order terms of the solution inner expansion in the vicinity of the critical layer are

$$\hat{\psi} \to \tfrac{1}{2} V'' A \hat{z}^2 + \partial_Y^{-1} \partial_T A + s\tfrac{1}{2} A^2 - r\hat{G}[A] + O(\epsilon^{1/2}), \tag{3.22}$$

the boundary condition at $\hat{z} = 0$ once again leading to (3.16) with *new* coefficients

$$r = \frac{V_\infty^2}{H_\infty} \frac{H'}{V''}\bigg|_{\xi=0}, \qquad s = \frac{V''}{H'}\bigg|_{\xi=0}. \tag{3.23}$$

3.3. *'Short-wave' and 'long-wave' asymptotics*

Thus we have derived a class of nonlinear evolution equations (3.16) describing spatially temporal (Y, T) dynamics of weakly nonlinear, weakly dispersive vorticity waves. The dependence on the cross-shore coordinate is given by the explicit expression (3.5) (the 'fast' part) and the boundary-value problem (3.18) (the 'slow' part). The coefficients in the equation are determined by bound values of 'fast' dependencies of the current and depth, while a slow depth dependence on the cross-shore coordinate is responsible for the specific dispersion. Different depth profiles outside a narrow coastal zone lead to very different forms of the operator G kernel and therefore of the dispersion law. Nevertheless despite a great variety of possible kernels of G there is a certain, not obvious, universality in the properties of all the variety of the nonlinear evolution equations belonging to this class.

We recall that a non-dimensional parameter μ, characterizing the ratio of the alongshore scale L to the cross-shore 'slow' scale d_{slow}

$$\mu = \frac{L}{d_{slow}} \tag{3.24}$$

was introduced above and taken to be of order unity when setting out the asymptotic procedure in §2. The asymptotic scheme also remains valid when this ratio is of a different order. It should be noted that within the adopted approach we are considering the evolution of *wide-band* wavetrains, and thus at the periphery of the spectral band in Fourier space necessarily have relations $\mu \gg 1$ and $\mu \ll 1$. Thus the question of the asymptotics for large and small μ is of true interest.

We will show explicitly that for any arbitrary depth profile with a scale much larger than a typical wavelength the dispersion closely corresponds to that involved in the well-known Benjamin–Ono (BO) equation (Benjamin 1967).

Let us suppose that

$$D = D(\mu X), \tag{3.25}$$

where μ is a small parameter $\mu \ll 1$.

Under assumption (3.25) one can look for a solution to (3.18) in the form of a power series in μ:

$$\phi_k(X) = \phi_{0k}(X) + \mu\phi_{1k}(X) + \dots . \tag{3.26}$$

Taking into account that under assumption (3.25) the depth cross-shore derivative becomes $\partial_X D = \mu D'$ and substituting (3.26) into (3.18*a*), one easily gets at the main order in μ the equation for $\phi_{0k}(X)$

$$\partial^2_{XX}\phi_{0k} - k^2\phi_{0k} = 0. \tag{3.27}$$

The only solution of (3.27) satisfying the boundary condition at infinity is the decaying exponent

$$\phi(X) = \exp\{-|k|X\} \tag{3.28}$$

that yields the corresponding kernel of the dispersion operator (3.17) with the accuracy of $O(\mu)$

$$\partial_X \ln \phi_k(X)\big|_{X=0} = -|k|. \tag{3.29}$$

The dispersion defined by the operator kernel (3.29) is obviously of the Benjamin–Ono type.

Unfortunately, we cannot establish any general result for the opposite case of very long vorticity waves (so that $\mu \gg 1$), nevertheless the straightforward analysis of some particular depth cross-shore dependencies is possible and is done below.

3.4. *Particular examples*

Boundary problem (3.18) specifying the integral kernel can be solved analytically for some large-scale cross-shore depth dependencies. Consider two particular examples of this kind in more detail.

3.4.1. *Exponential profile*

Let the dependence of the depth on the slow cross-shore variable be of the form

$$D = \exp\{qX\}, \tag{3.30}$$

where q is a positive constant. Then the integral kernel defined by the solution of (3.18) becomes

$$\left.\frac{\partial_X\phi_k}{\phi_k}\right|_{X=0} = \tfrac{1}{2}\left(q - \left(q^2 + 4k^2\right)^{1/2}\right). \tag{3.31}$$

The kernel (3.31) demonstrates explicitly an intermediate character of our equation:

$$\left.\frac{\partial_X\phi_k}{\phi_k}\right|_{X=0} \to \begin{cases} -|k| & \text{for} \quad k \gg q \qquad \text{(BO limit)}, \\ -\dfrac{k^2}{q} & \text{for} \quad k \ll q \qquad \text{(KdV limit)}. \end{cases} \tag{3.32}$$

The exponential depth profile (3.30) apart from being a good illustrative example is said to be a good model of the East-Australian coast (see LeBlond & Mysak 1979). Some solutions to the evolution equation with kernel (3.31) are discussed below.

3.4.2. *Power-law profile*

Consider now the large-scale cross-shore depth dependence to be of the form

$$D(X) = (1+X)^{2m}. \tag{3.33}$$

In this case (3.18*a*) could be resolved in terms of McDonald functions, the solution subject to (3.18*b*) being of the form

$$\phi_k(X) = (1+X)^{m+1/2} K_{m+1/2}[k(1+X)]. \tag{3.34}$$

Using the well-known property of McDonald functions (see Olver 1974)

$$zK'_\nu(z) = -\nu K_\nu(z) - zK_{\nu-1}(z), \tag{3.35}$$

we get the dispersion kernel in the form

$$Q(k) = -|k| \frac{K_{m-1/2}(|k|)}{K_{m+1/2}(|k|)}. \tag{3.36}$$

Small-k asymptotics of the kernel depend on the value of parameter m and lead to totally different dispersion types at the small-k periphery of the nonlinear wavetrain.

For $m > 1/2$ by using an asymptotic expansion of the McDonald function of positive and real index for a small argument (see Olver 1974) we obtain

$$Q(k) \to -\frac{k^2}{2m-1} \qquad \text{for} \quad |k| \ll 1, \tag{3.37}$$

which corresponds to the KdV-type dispersion. Thus both exponential and power-law profiles with exponent greater than unity (we will call them 'steep' hereinafter) lead to a nonlinear evolution equation for vorticity waves which is 'intermediate' between BO and KdV equations in the sense that the dispersion at the peripheries of the *wide-band* wavetrains studied here is of KdV (for $\mu \gg 1$) and BO (for $\mu \ll 1$) types†.

Power depth profiles with exponents equal to unity and less (we will call them 'flat') lead to dispersion kernels which exhibit quite different behaviour at the small-k limit and hence corresponding evolution equations do not tend to KdV as k tends to zero. First, let the power exponent $m = 1/2$, then from (3.36) we obtain in the small-k limit

$$Q(k) \to k^2 \ln|k| \qquad \text{for} \quad |k| \ll 1. \tag{3.38}$$

When m is less than $1/2$ we use another property of McDonald functions, that for real $x > 0$ there is a symmetry with respect to the index: $K_{-\nu}(x) = K_\nu(x)$, and get

$$Q(k) \to -2\frac{\Gamma(\frac{1}{2}-m)}{\Gamma(\frac{1}{2}+m)} \left(\frac{|k|}{2}\right)^{2m+1} \qquad \text{for} \quad |k| \ll 1, \tag{3.39}$$

where $\Gamma(z)$ is gamma-function.

Some properties of the steady solitary solutions of (3.16) implied by the particular form of the small-k asymptotic of the dispersion kernel will be studied below.

3.5. *Generalization of the results for currents with two cross-shore scales*

The assumption made earlier of the mean current not having a slow space scale greatly simplifies the study of vorticity wave nonlinear dynamics, especially the analysis of particular models. However, most oceanic currents do possess different cross-shore

† We note that the so-called 'intermediate long-wave equation' first derived by Joseph (1977) does not belong to the class (3.16).

scales, so taking this fact into account is important. Let us presume that the current velocity as well as the depth depend on both 'fast' and 'slow' scales, that is,

$$h = H(\xi)D(X), \qquad V = V(\xi, X). \tag{3.40}$$

Performing the necessary calculations is a bit more tedious than in the previous case but the final results for phase speed and slow-time evolution do not differ much:

$$c = V(0), \tag{3.41}$$

$$\partial_T A + s\,A\partial_Y A - r\,\mathsf{G}[\partial_Y A] = 0, \quad r = \frac{V^2(\infty,0)}{H_\infty}\frac{H}{V'}\bigg|_{\xi,X=0}, \qquad s = \frac{V'}{H}\bigg|_{\xi,X=0}, \tag{3.42}$$

with the same dispersion operator (3.17). The difference lies in the boundary problem specifying the operator G kernel. It becomes

$$V_\infty\left(\frac{\phi_k'}{D}\right)' - \left(\frac{V_\infty'}{D}\right)'\phi_k - V_\infty k^2\frac{\phi_k}{D} = 0, \tag{3.43a}$$

$$\phi_k(X) \to 0 \qquad \text{as} \qquad X \to \infty, \tag{3.43b}$$

where now a prime designates the derivative with respect to X and

$$V_\infty = V(\infty, X).$$

Evidently, to get explicit solutions and corresponding operator kernels from (3.43) is much more difficult than from (3.18), which was the prime reason for getting rid of the slow-scale velocity dependence of the mean flow. The main justification of doing so is that we do not expect qualitative differences compared to the much simpler case analysed above.

4. Steady solitary-wave solutions

As shown above, the nonlinear evolution equation (3.16) with the dispersion operator generated by (3.18) represents an intermediate case between BO and KdV equations for the 'steep' large-scale cross-shore depth profile. The most important solutions to both BO and KdV equations playing the key role in the evolution of a wide class of initial conditions are steady solitary waves, i.e. localized perturbations of a permanent form propagating with a constant speed. One is tempted to investigate whether (3.16) possesses steady solutions of a solitary-wave type and, if yes, what common features these solutions have and how the specific dispersion created by a given depth profile is revealed in the solution. Here we shall investigate these questions both analytically and numerically.

Let us look for a steady solution of (3.16) in the form

$$A(Y, T) = A(Y - UT), \tag{4.1}$$

where U is the constant speed of a solitary wave. After substituting (4.1) into (3.16) and integrating with respect to $\eta = Y - UT$ we get the nonlinear equation for the alongshore dependence of the steady wave amplitude in the frame of reference moving with constant speed U:

$$UA + r\mathsf{G}[A] - s\tfrac{1}{2}A^2 = 0, \tag{4.2}$$

where $\mathsf{G}[f]$ is the pseudo-differential operator (3.17). We recall that r and s are positive constant coefficients, which without loss of generality can be put equal to, say, unity by means of a renormalization.

To solve (4.2) analytically is by no means easy even for the simplest forms of the dispersion operator G: at least we are not aware of any analytical solutions of equations of this type apart from the already well-known BO, KdV and Joseph equations. Thus one is forced to treat this problem numerically, although some important asymptotic properties of the solutions can be established analytically.

For the numerical treatment we used the program kindly provided to us by Y. A. Stepanyants and modified especially for our problem by S. Y. Annenkov. In its turn it is based on an idea for a numerical algorithm developed by V. I. Petviashvilli (see e.g. Petviashvilli & Pokhotelov 1992 for the numerical search of solitary solutions in a quite different context). This approach proved to be best suited to solving equations with a power-law nonlinearity and an arbitrary dispersion. Below we sketch the main points of the procedure.

First, performing a Fourier transform of (4.2) yields

$$(U + rQ(k))\tilde{A} = s\tfrac{1}{2}\tilde{A}^2, \tag{4.3}$$

where $Q(k)$ is the kernel of the dispersion operator G: $Q(k) = \partial_X \ln \phi_k$ at $X = 0$, and the tilde denotes the Fourier components of the functions.

As the derivative of the large-scale cross-shore-dependent part of the streamfunction ϕ_k at point $X = 0$ is always negative for a 'steep' profile the kernel $Q(k)$ is also negative for all wavenumbers. This fact together with the right-hand side of (4.3) being definitely positive makes one conclude that $U < 0$ and therefore $\tilde{A} < 0$, $\forall Y$. So the steady solitary waves under consideration are always moving upstream and have negative amplitude. Rescaling the wavenumber and the amplitude of the streamfunction

$$K = \frac{r}{|U|}k, \qquad \tilde{B} = \frac{s}{U}\tilde{A}, \tag{4.4}$$

we get a simpler non-dimensional equation in terms of $\tilde{B}$:

$$\tilde{B}(1 + |\tilde{Q}(K)|) = \tfrac{1}{2}\tilde{B}^2, \tag{4.5}$$

where the new renormalized kernel $\tilde{Q}(K)$

$$\tilde{Q}(K) = \tfrac{1}{2}\left(p - (p^2 + 4K^2)^{1/2}\right) \tag{4.6}$$

depends upon only one non-dimensional parameter, $p = qr/|U|$. Now the function $\tilde{B}$ can be found from (4.5) through the iteration scheme

$$\tilde{B}_{n+1} = \frac{M^2 \tilde{B}^2{}_n}{2(1 + |\tilde{Q}(K)|)}, \tag{4.7}$$

where a multiplier

$$M^2 = 2\frac{\int (1 + |\tilde{Q}(K)|)(\tilde{B}_n)^2 \, \mathrm{d}K}{\int \tilde{B}_n \tilde{B}_n^2 \, \mathrm{d}K} \tag{4.8}$$

is used to stabilize the convergence of the iteration procedure. It is easy to see that (4.8) equals unity when computed for the exact solution of (4.6).

Numerical computations were performed for the equation with the dispersion kernel (3.31) (corresponding to the exponential depth dependence) until the difference between the multiplier and unity became smaller than 10^{-4}. After the condition

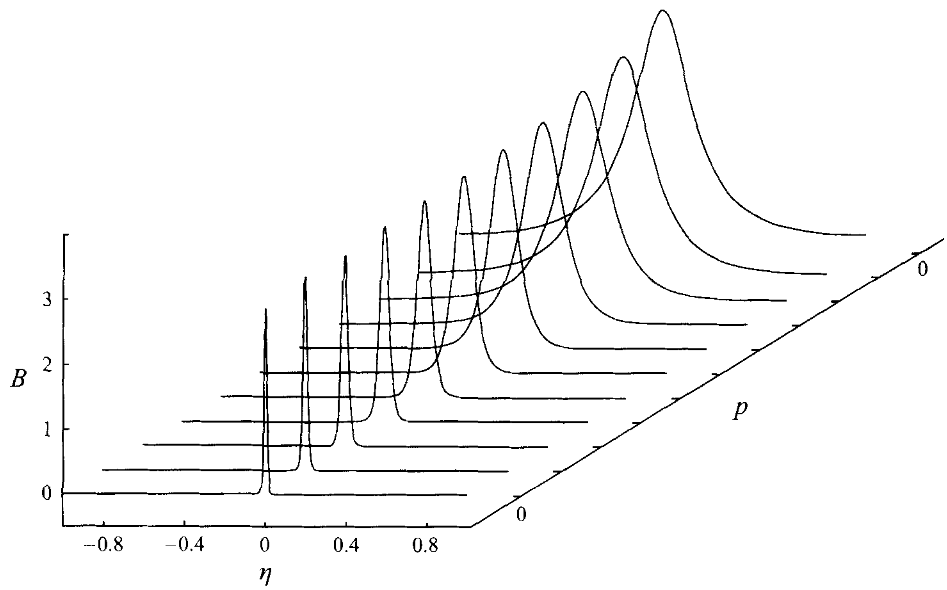

FIGURE 2. Soliton solutions $B(\eta)$ *vs* the scale p^{-1} of the exponential continental slope.

$|1-M| < 10^{-4}$ was fulfilled the iteration process was stopped. Each time, after 20–30 iterations the process converged to the universal solution regardless of the specific form of the initial amplitude distribution taken as the starting function B_0.

The resulting solutions depending on the non-dimensional shelf-steepness scale p are plotted in figure 2. Unfortunately no explicit relationship between the size, velocity and amplitude of the solitary-wave solutions of (3.16) was found similar to the well-known ones for the KdV and BO equations. Still one can easily see from the plot that the solitons of the BO and KdV equations represent the limiting cases for those of (4.2) when the parameter p is changing from zero to infinity. Moreover, the periphery of all computed solitary waves, the 'tails', look close to the KdV solitons, a manifestation of the universal asymptotic 'intermediate' properties of the evolution equation discussed above. The latter fact will be shown analytically below for the whole class of evolution equations with dispersion determined by the 'steep' depth profiles.

From the asymptotic analysis of the dispersion operator kernel one has to conclude that (3.16) represents an intermediate case between BO and KdV equations for the 'steep' depth profiles. Moreover, essential information can be extracted from the asymptotic features of the dispersion kernel $Q(k)$ determined by (3.18) in the limit of small k, concerning the behaviour of the steady solitary solution $A(\eta)$ at large values of $|\eta|$. We claim that the asymptotics of $A(\eta)$ at large values of $|\eta|$ coincide with the KdV soliton asymptotics, i.e. tails of the solitary wave decay exponentially. To prove this statement consider (4.3) in Fourier space. As shown above, in the limit of $k \to 0$ an arbitrary dispersion kernel $Q(k)$ turns out to be a KdV kernel, namely $const * k^2$, the constant being equal to $-q^{-1}$ for the kernel (3.31) and to $-(2m-1)^{-1}$ for the kernel (3.37). A Fourier transform of the square of the function represents a convolution

$$(A^2)_k = \int_{-\infty}^{+\infty} A_\kappa A_{k-\kappa}\, \mathrm{d}\kappa. \tag{4.9}$$

The convolution (4.9) can be approximated in the small-k limit by the value $(A^2)_0$ with error $O(k^2)$ due to the symmetry of the solution. Thus in this limit the solution of the nonlinear integral equation (4.3) is

$$A_k \approx -\frac{s}{2|U|}\frac{(A^2)_0}{1+\gamma^2k^2}. \tag{4.10}$$

One can recognize in (4.10) the small-k asymptotics of the Fourier transform of a KdV soliton or check this directly by performing the corresponding inverse Fourier transform, which yields the result claimed above:

$$A \sim \exp(-|\eta|/\gamma), \tag{4.11a}$$

where

$$\gamma^2 = \begin{cases} \dfrac{r}{q|U|} & \text{for exponential depth} \\ \dfrac{r}{(2m-1)|U|} & \text{for 'steep' power depth .} \end{cases} \tag{4.11b}$$

This result could also be directly seen from the fact that in the small-k limit (4.5) becomes the stationary KdV equation. The result is quite natural as the behaviour of a localized smooth function at large values of its argument is determined by the form of its spectrum at small k only, which can be easily extracted from the Fourier form of the equation. Thus we have proved the universal character of exponential decay of all solitary-wave solutions for the 'steep' depth dependence.

The fact of exponential decay just established is of prime importance. In particular, interactions among solitary waves, in fact the solution to a non-steady problem for a wide class of initial data, can often be described as a specific interaction between the particles, the main features of the interaction being entirely determined by the behaviour of their 'tails' (e.g. Gorshkov & Ostrovsky 1981). Precise knowledge of the tail asymptotics could allow one to develop a procedure for field data processing to extract these patterns from the noisy records.

5. Effect of the Earth's rotation on the vorticity wave dynamics

Wave-like motions considered to be vorticity waves have characteristic frequencies much higher than the Coriolis parameter f determining the frequency scale of the continental shelf waves. It is this fact that permits one to separate these two classes of waves in the coastal zone that are similar to each other in many other aspects. Mathematically we separate them by assuming the Coriolis terms in the full shallow-water equations to be negligibly small in comparison with the inertial terms involving the mean alongshore current. So the Earth's rotation does not affect the linear dynamics of vorticity waves. But (2.7) includes both linear and nonlinear terms on the right-hand side. For small-amplitude vorticity waves these nonlinear terms are also small and thus might be comparable with the omitted Coriolis terms.

Consider this effect in detail. If retained, the Coriolis term in (2.7) would be of the form

$$\text{Coriolis term} = -f\frac{d}{V_0}\left(\frac{1}{h}\right)_x \partial_y\psi, \tag{5.1}$$

where f is the Coriolis parameter, d and V_0 are the mean current cross-shore scale and velocity respectively. If the coefficient fdV_0^{-1} is of the same order, $O(\epsilon)$, as the velocity perturbation magnitude, the term (5.1) does not influence the linear dynamics of vorticity waves but is of the same order as the nonlinear inertial terms and so does affect the nonlinear dynamics.

Assume that

$$f\frac{d}{V_0} = \epsilon f\frac{L}{V_0} = \epsilon F, \tag{5.2}$$

where F is a constant of $O(1)$ and the small parameter ϵ was defined earlier. Taking into account (5.2) and returning to our notation of (2.13)–(2.15) we obtain instead of (2.7) an equation governing the nonlinear dynamics of vorticity waves in a 'slowly' rotating fluid:

$$(V\partial_Y + \epsilon\partial_T)\left(\frac{\psi_x}{h}\right)_x - \left(\frac{V'}{h}\right)_x \psi_Y = -\left(\frac{\psi_x}{h}\frac{\psi_{Yx}}{h} - \frac{\psi_Y}{h}\left(\frac{\psi_x}{h}\right)_x\right)_x$$
$$+ \epsilon F\left(\frac{1}{h}\right)_x \psi_Y - \epsilon^2 V\partial^3_{YYY}\frac{\psi}{h} + \epsilon^2\left(\frac{\psi_y}{h}\left(\frac{\psi_y}{h}\right)_x - \frac{\psi_x}{h}\frac{\psi_{yy}}{h}\right)_Y. \tag{5.3}$$

As the Coriolis force does not affect the linear vorticity wave dynamics, the first-order approximation represents an identity. But at the second order the result slightly differs from (3.9) and has the form

$$\partial_Y\left(V\left(\frac{\psi_2'}{h}\right)' - \left(\frac{V'}{h}\right)'\psi_2\right) = -(\phi * A_T)\left(\frac{V'}{H}\right)' + \left(\frac{1}{H}\right)' FV(\phi * A_Y)$$
$$-\left(\frac{V^2}{H}\right)'(\phi_X * A_Y) + D^{-1}\left(\frac{V^2}{H}\left(\frac{V'}{VH}\right)'\right)'(\phi * A)(\phi * A_Y). \tag{5.4}$$

Considering (5.4) as an ordinary differential equation with ξ as the independent variable we integrate it, the solution subject to (2.16*b, c*) being

$$\partial_Y\psi_2 = (\phi * A_T) + (\phi_X * A_Y)V\int_\infty^\xi\left(\frac{V_\infty^2}{V^2(\xi)}\frac{H(\xi)}{H_\infty} - 1\right)\mathrm{d}\xi$$
$$-(\phi * A_Y)FV\int_\infty^\xi\frac{H}{V^2}\left(\int_\zeta^\infty V\left(\frac{1}{H}\right)'\mathrm{d}\tau\right)\mathrm{d}\zeta + D^{-1}\frac{V'}{H}(\phi * A)(\phi * A_Y). \tag{5.5}$$

The third-order approximation in powers of ϵ for (5.3) includes as in § 3 both secular and non-secular terms, so the requirement of solution regularity yields the necessary equation for the proper slow cross-shore dependence. Thus the task is accomplished by the same Fourier transformation procedure as when deriving (3.16)–(3.18) to yield the nonlinear evolution equation for a long, finite-amplitude vorticity waves in a slowly rotating fluid:

$$\left.\begin{aligned} &\partial_T A + sA\partial_Y A - r\mathrm{G}[\partial_Y A] = -f^*\partial_Y A, \\ &f^* = F\frac{H}{V'}\bigg|_{\xi=0}\int_0^\infty V\left(\frac{1}{H}\right)'\mathrm{d}\xi, \end{aligned}\right\} \tag{5.6}$$

where G[*f*] is pseudo-differential operator of the type (3.17). Equation (5.6) reduces by a Galilean transform to the same form as in a non-rotating fluid (see (3.16)) with one essential difference: the kernel of the dispersion operator is now determined by a modified boundary problem (cf. (3.17)–(3.18))

$$\left(\frac{\partial_X \phi_k}{D}\right)_X - \frac{F}{V_\infty}\left(\frac{1}{D}\right)_X \phi_k - k^2\frac{\phi_k}{D} = 0; \tag{5.7a}$$

$$\phi_k \to 0 \quad \text{as} \quad X \to \infty. \tag{5.7b}$$

As one can easily see (5.7a) exactly corresponds to the equation governing continental-shelf wave dynamics, the magnitude of the mean alongshore current velocity in the deep ocean playing the role of the wave phase speed.

Thus the effect of a relatively weak Coriolis force on the vorticity wave nonlinear dynamics manifests itself in two ways: first and most important, it modifies their dispersion through the change in the operator G kernel; second, it adds a constant of order ϵ to the phase velocity in the long-wave limit. Note that the latter appears only when a 'fast' depth variability is present, while the modification of the integral kernel is prescribed by a 'slow' depth cross-shore dependence only. The most striking phenomenon caused by the Earth's rotation is the *linear resonance* between the vorticity and shelf waves that occurs at some special depth profiles and mean current velocity magnitudes. Its essence can be ascertained as follows. A solution of (3.18) evidently cannot be zero at $X = 0$ for any depth profile, in contrast to the solution of (5.7). Indeed, consider the boundary value problem for continental-shelf waves propagating alongshore in a coastal zone with the same cross-shore depth dependence $D(X)$ in the absence of any mean current:

$$\left(\frac{\phi_n'}{D}\right)' - \frac{F}{c_n}\left(\frac{1}{D}\right)' \phi_n - k^2\frac{\phi_n}{D} = 0, \tag{5.8a}$$

$$\phi_n\Big|_{X=0} = 0, \tag{5.8b}$$

$$\phi_n \to 0 \quad \text{as} \quad X \to \infty. \tag{5.8c}$$

Suppose further that this problem has eigensolutions corresponding to continental-shelf waves propagating alongshore with the phase speed $c_n = c_n(k)$ and having cross-shore eigenfunctions $\phi_n(X)$. If for a wavenumber $k = k_*$ the phase speed of a continental-shelf wave mode is the same as V_∞ from (5.7) then the corresponding eigenfunction ϕ_n of (5.8a–c) is identical to the solution ϕ_k of (5.7a). But because of (5.8b) ϕ_k turns out to be zero at the point $X = 0$ and, its derivative not necessarily being zero, the integral kernel, as one can easily see, becomes infinite at the point $k = k_*$ and the dispersion operator G becomes singular. The equality of a continental-shelf wave-mode phase speed and the mean current velocity at the deep ocean at some wavenumber k_* value means the matching of the phase speeds of a long vorticity wave and a particular continental-shelf wave mode (the Doppler shift must be taken into account). As is common for linear resonance problems a special although straightforward analysis should be carried out in the neighbourhood of the 'intersection' point (see e.g. Craik 1985). The linear interaction of resonant modes results in dispersion enhancement and splitting of the dispersion curves which, in turn, leads either to the so-called 'change of identities' or to linear instability, depending on the signs of the energies of the interacting waves. The analysis for some model depth cross-shore profiles permitting the analytic solution of (5.8a) did not exhibit any linear instability though the possibility of its existence still remains. This strong linear interaction and dispersion enhancement results in the fact that the dispersion operator G becomes singular and our asymptotic procedure as well as (5.6) lose their validity. Another result of the linear resonance is likely to be the radiation of resonant continental-shelf waves by the running vorticity waves and therefore their damping through a mechanism similar to the Landau damping in plasmas. However, the problem of the vorticity wave nonlinear dynamics in the nearshore is principally two-dimensional and as the continental-shelf waves phase speed is bounded from above the process of linear interaction which breaks the proposed asymptotic scheme does not occur for a sufficiently strong mean flow (such that inequality $c_{max} < V_\infty$ holds for all possible continental-shelf wave wavenumber values). This being the case, (5.6) is valid and the dispersion operator G is regular.

6. Resonant forcing by coastline inhomogeneity

Until now throughout the paper we supposed the coastline to be straight. This idealization, while giving an obvious gain in simplicity and clarity of consideration, at the same time narrows considerably the applicability of the approach to realistic situations. Moreover some qualitative effects are lost. Fortunately the effect of coastline inhomogeneity can be easily taken into account within the framework of the approach developed if the inhomogeneity were small enough and varied in the alongshore direction on a scale comparable with the typical wavelength L. The study of the generation of vorticity waves due to this effect is the subject of the present §.

To make our assumptions on the inhomogeneity features explicit, let us define a coastline $x = x_0(y)$ in the same Cartesian frame (x, y) in the form

$$x = x_0(y) \equiv \alpha Q(y), \tag{6.1}$$

where $\alpha \ll 1$ is a small parameter characterizing the magnitude of the coastline departure from the straight line $x = 0$. We recall that (2.8a) requires the mass flux normal to the coast to be zero. For the long vorticity waves considered above (which have zero phase speed at the main order in ϵ and all the scaling and notation preserved) this yields

$$\left(\psi_Y + \alpha Q_Y H V + \alpha Q_Y \psi'\right) = 0 \qquad \text{at} \qquad \xi = \alpha Q(Y) = 0. \tag{6.2}$$

As the parameter α is presumed to be small, (6.2) can be expanded in a Taylor series with respect to αQ in the vicinity of the line $\xi = 0$. The calculation results in (at $\xi = 0$)

$$\psi_Y + \alpha Q_Y H V + \alpha Q \psi'_Y + \alpha Q_Y \psi' + \alpha^2 Q Q_Y (HV)' = O\left(\alpha^3\right). \tag{6.3}$$

The boundary condition (6.3) should be used in the nonlinear boundary value problem (2.16), (2.7) instead of the original condition on the straight coastline. To proceed further one should assume some balance between the parameters α and ϵ. Under the earlier assumption $V(0) = 0$ the second term on the left-hand side of (6.3) disappears and we have to suppose the balance $\alpha = \epsilon$. Thus the effect of coastline inhomogeneity on vorticity waves is weaker than on resonant continental-shelf waves where the balance is $\alpha = \epsilon^2$ (Grimshaw 1987). On substituting this scaling and the streamfunction in the asymptotic form (3.1) into (6.3) we get at the main (first and second) orders in ϵ

$$\psi_{1Y}\Big|_{\xi=0} = 0, \tag{6.4a}$$

$$\psi_{2Y} = -\tfrac{1}{2}\left(Q^2\right)_Y HV' - \left(Q\psi'_1\right)_Y \qquad \text{at} \qquad \xi = 0. \tag{6.4b}$$

The boundary condition (6.4a) applied to (3.5) yields once again (3.7), (3.8) ensuring that the weakly dispersive vorticity waves remain resonant with the coast when the mean current speed at the coast equals zero. Thus the procedure we used is valid. Then we can apply (6.4b) to the first-order nonlinear solution (3.10) to obtain instead of (3.16) a forced nonlinear evolution equation for long, finite-amplitude vorticity waves:

$$\left.\begin{aligned} &\partial_T A + sA\partial_Y A - r\mathrm{G}[\partial_Y A] + V'|_{\xi=0}(QA)_Y + \mathcal{G} = 0, \\ &\mathcal{G}(Y) \equiv \left(\tfrac{1}{2}Q^2\right)_Y HV'\Big|_{\xi=0}, \end{aligned}\right\} \qquad (6.5)$$

which differs from (3.16) in the presence of the forcing terms. We stress that $\mathrm{G}[f]$ is again the pseudo-differential operator (3.17) determined by the solution of the same boundary problem (3.18).

If both $H(0)$ and $V'(0)$ equal zero the terms of order α in (6.3) are also zero and one has to expand (6.2) to the next order in α. Finally we get instead of (6.3)

$$\psi_Y + \tfrac{1}{6}\alpha^4 Q^3 Q_Y (HV)''' + \tfrac{1}{2}\alpha^2 Q^2 \psi_Y'' + \alpha^2 Q Q_Y \psi'' = O\left(\alpha^5\right). \qquad (6.6)$$

Now we have to assume the balance $\alpha = \epsilon^{1/2}$. This immediately leads to the modified 'forced' evolution equation instead of (6.5):

$$\left.\begin{aligned} &\partial_T A + sA\partial_Y A - r\hat{G}[\partial_Y A] + \tfrac{1}{2}V''\Big|_{\xi=0}\left(Q^2 A\right)_Y + \mathcal{G} = 0; \\ &\mathcal{G}(Y) \equiv \left(\tfrac{1}{24}Q^4\right)_Y H'V''\Big|_{\xi=0}. \end{aligned}\right\} \qquad (6.7)$$

Thus we have shown that even a comparatively small inhomogeneity of the coastline provides an effective mechanism of vorticity wave generation, the process and vorticity wave nonlinear dynamics being governed by (6.5) or (6.7). The same mechanism of vorticity wave resonant forcing occurs owing to the alongshore non-uniformity of the bottom and also to the mean wind stress, if present. The only difference lies in the specific expressions for the forcing term $\mathcal{G}$, their derivation within the asymptotic procedure developed being straightforward. This phenomenon is quite similar to continental-shelf wave generation by the same forces considered by Grimshaw (1987) though the effective forcing of vorticity waves due to zero mean current at the shore requires stronger inhomogeneity of the coastline or the bottom than that of resonant continental-shelf waves.

It is beyond the scope of the present work to deduce immediate implications of the derived equation. The study of the forced nonlinear evolution equations has become a subject of increasing interest in recent years; the advances in this area are mostly related with the series of works by Grimshaw (see e.g. Grimshaw, Pelinovsky & Xian 1994) and up to now have been mainly concerned with the forced KdV equation. At present our forced equation just provides a promising framework for further investigations.

7. Discussion

First we briefly summarize the main results of our study and discuss the key assumptions, limitations and perspectives of the approaches developed.

The main aim of the paper was to draw attention to a new (in the context of geophysical fluid dynamics) type of wave motions and to develop a relatively simple mathematical model describing their basic properties. These motions are specific vorticity waves, expected to occur in the coastal zone and distinct in scales and properties from the already-known continental-shelf waves and shear or vorticity waves of Oltman-Shay *et al.* 1989.

From the observational point of view vorticity waves manifest themselves mainly as variations of an alongshore velocity of the mean currents. Typical periods are a few hours, whereas typical spatial scales exceed greatly the cross-shore scale of the basic current and can be in the range from a few dozens to a hundred kilometres. We are unaware of any field observations where these motions have been identified. There are some practical difficulties in detecting these motions by traditional oceanographic means as measurements made at one point do not allow one to distinguish vorticity waves from other types of motions (first of all, internal waves) belonging to the same frequency range. To select vorticity waves for certain one needs a spatial array of sensors with a base of a hundred kilometres located along the periphery of a suitable current. However fast progress in remote sensing techniques promises to make the problem of detecting vorticity waves easier. We hope that the mathematical model developed above will help in identifying these motions.

The model reduces the description of weakly nonlinear vorticity wave dynamics to a single *one-dimensional* evolution equation. This gives a substantial gain in simplicity: the new evolution equation is obviously much more simple than the original two-dimensional nonlinear boundary problem. The equation is of the generalized Benjamin–Ono type, the pseudo-differential dispersion operator being specific for each combination of bottom and current profiles. The derived equations form a wide new class and have an important common property: they all tend to the BO limit as the wave scale becomes short in comparison to the shelf-slope variation characteristic scale. For a family of model profiles the equations are intermediate between BO and KdV equations. Their solitary-wave solutions for various depth profiles decay exponentially. It is worth mentioning that the latter fact allows one to apply the previously developed specific perturbation technique in order to describe the interaction among the solitary waves despite not knowing precisely their shape. We note that a special study of the solutions and properties of these equations is still to be done; it was not among the prime priorities of this work.

We would like to draw attention to the point that within the framework of our approach the generation of vorticity waves by a coastline inhomogeneity or/and alongshore depth variations is naturally described by the forced evolution equations and that this mechanism of generation is rather effective and robust even for small inhomogeneities of the coastline or bottom profile. It holds even when viscous effects are important and *all* vorticity wave modes are decaying.

All the above-mentioned results were derived using the following key assumptions:

(i) the perturbations of the currents are weakly nonlinear waves;

(ii) vorticity waves are long in comparison with the cross-current typical scale;

(iii) the basic current is characterized by a monotonic profile of potential vorticity;

(iv) viscous effects are neglected.

How important are they?

The first two assumptions are unavoidable when describing vorticity wave dynamics via nonlinear evolution equations. Only long-wave perturbations persist for sufficiently long times and could be treated as waves.

The third assumption is necessary to ensure the stability of the basic current and the absence of discrete spectrum modes in the linearized inviscid boundary value problem. It is this assumption that allows us to reduce the description of the dynamics of a weakly nonlinear wave field to the description of the evolution of a single 'quasi-mode'; it should also be noted that under this assumption the long-wave perturbations of the basic flow dominate the weakly nonlinear regime. Thus the evolution equation describes the most important range of the vorticity wave field. If we were to remove this restriction and try to keep the approach based on the evolution equations, we would face a number of questions, some of which could be resolved. First we note that the evolution equation holds, but now we cannot claim that it describes the most important part of the field variability. The range of scales with wavelengths of order of the cross-current typical scale is most likely to be dominant and therefore, as a rule, to be the most interesting. For the perturbations of these scales only envelope-type equations could be applied. However interest depends entirely on the physical context, for example to study the role of vorticity waves in the sedimentation processes one should focus attention on waves having almost zero velocity with respect to the shore, i.e. the waves governed by the derived evolution equations.

For the situations under consideration the bottom friction, the main viscous effect, could be easily incorporated into the evolution equation provided it is of the same order as the nonlinearity and the dispersion. Although we do not expect new qualitative effects due to bottom friction, the quantitative differences might be quite noticeable and the role of this effect in vorticity wave nonlinear dynamics should be carefully examined.

The vorticity waves are of undoubted interest in themselves. One of the features which makes them especially interesting is their nearly zero, with respect to the shore, phase and group velocity. One may expect them to have an important role in the process of sedimentation even when their contribution in the nearshore variability is not dominant. We consider studies of the mechanisms of vorticity wave contribution to sedimentation to be the most promising direction of further research in this field. Another characteristic of vorticity waves that is worth mentioning is related to the strongly nonlinear motions they generate in the vicinity of the critical layer, i.e. in the immediate vicinity of the shore. Although these strongly nonlinear vortical motions cannot be described by our model (we just have shown that they do not affect the waves at the leading order) we can predict some essential parameters of these induced vortical motions. Such vortices, being localized in the immediate vicinity of the shoreline, could contribute fundamentally to the processes of mixing and diffusion in this important zone. The situation somewhat resembles that of Pedley & Stephanoff (1985) where intense vortices, which were detected experimentally in the near-wall boundary layers in Poiseuille flow in a channel, were not described by the theory. However vorticity waves described by their linear model provided some characteristics of the induced vortices. A more distant analogy would be water waves with breakers. Although unable to describe the breakers (vortices) we nevertheless know that they are attached to the crests and on this basis can predict their propagation.

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